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# ON CONSTRUCTION OF FRACTIONAL REPLICATES AND ON ALIASING SCHEMES

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ABSTRACT

A generalized method of constructing irregular fractional replicates from a complete factorial is developed and alias schemes in some main effect fractional replicates are presented in this paper. Special reference is made to the construction of saturated fractional replicates for a set of main effect parameters. The method of construction involves a special ordering of the treatment observations and of the single degree of freedom parameter contrasts. Prior to presentation of the method, a Kronecker product representation is given for the design matrices and corresponding orthogonal arrays are investigated. Also, an invariant property of the information matrices of the main effect fractional replicates is investigated and a property of the aliasing structure is discussed. In the last section, we present some confounded patterns, or aliasing schemes, in some main effect fractional replicates. Special reference is made to the $2^2$, $2^3$, $2^4$ and $3^3$ factorials.

# ON CONSTRUCTION OF FRACTIONAL REPLICATES AND ON ALIASING SCHEMES[1]

U. B. Paik[2] and W. T. Federer[3]

## 0. INTRODUCTION AND SUMMARY

A generalized method of constructing irregular fractional replicates from a complete factorial is developed and alias schemes in some main effect fractional replicates are presented in this paper. Special reference is made to the construction of saturated fractional replicates for a set of main effect parameters. The method of construction involves a special ordering of the treatment observations and of the single degree of freedom parameter contrasts. Prior to presentation of the method, a Kronecker product representation is given for the design matrix of $s^n$-factorial composed of linear contrasts and some relationships between the design matrices and corresponding orthogonal arrays are investigated. Also, an invariant property of the information matrices of the main effect fractional replicates is investigated and a property of the aliasing structure is discussed. In the last section, we present some confounded patterns, or aliasing schemes, in some main effect fractional replicates. Special reference is made to the $2^2$, $2^3$, $2^4$, and $3^3$ factorials.

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## 1. BASIC NOTATIONS AND STATISTICAL MODEL

In an $s^n$-factorial system ($s$ is a prime number), the space of treatment combinations, $Z$, is represented by the set $Z = \{(i_1, i_2, \ldots, i_n) : i_h = 0, 1, \ldots, s-1$ for all $h = 1, 2, \ldots, n\}$ which contains $s^n$ points, say $N = s^n$. A standard ordering of points in $Z$ is given by the relationship between the coordinate of a point $z_v = (i_1, i_2, \ldots, i_n)$, $v = 0, 1, \ldots, N-1$, and the order subscript

$$v = \sum_{h=1}^{n} i_h s^{n-h} . \tag{1.1}$$

The addition operator $+$ between any two treatment combinations $z_v$ and $z_{v'}$ is defined as follows: if $z_v = (i_1, i_2, \ldots, i_n)$ and $z_{v'} = (i'_1, i'_2, \ldots, i'_n)$ then $z_{v''} = z_v + z_{v'} = (i''_1, i''_2, \ldots, i''_n)$, where $i''_h = i_h + i'_h$, mod $s$, for all $h = 1, 2, \ldots, n$. It follows immediately that the set $Z$ is a group with respect to operator $+$. We denote by $\alpha z_v$, $\alpha = 0, 1, \ldots, s-1$, the addition of $z_v$ itself $\alpha$-times, i.e., $\alpha z_v = (\alpha i_1, \alpha i_2, \ldots, \alpha i_n) = (i'_1, i'_2, \ldots, i'_n)$, mod $s$.

The expected value of the random vector $y(Z)$ associated with the space of treatment combinations $Z$ is given by

$$E[y(Z)] = X\underline{B} , \tag{1.2}$$

where $X$ is an $N \times N$ orthogonal matrix in the sense that $X'X$ is a diagonal matrix, $\underline{B}$ is the $N \times 1$ column vector of single degree of freedom parameters, $\beta_0, \beta_1, \ldots, \beta_{N-1}$, and $y(Z)$ is the $N \times 1$ column vector with covariance matrix $\sigma^2 I$. The parameters $\beta_u$ have the usual interpretation of main effects and interactions of $n$ factors. We distinguish between linear effects, quadratic effects, and effects of higher order. (Note: Any orthogonal set of contrasts may be utilized but we have arbitrarily selected the polynomial set.) We also distinguish between linear by linear interactions, linear by quadratic interactions, etc. We further describe the structure of the $s^n = N$ parameters, $\beta_u$, $u = 0, 1, \ldots, N-1$, by considering the space $B$ of $N$ points where $B = \{B(\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_h = 0, 1, \ldots, s-1$ for all $h = 1, 2, \ldots, n\}$. The correspondence between the parameters $\beta_u$ and the points of $B$ is given by the order relation specified by $u = \sum_{h=1}^{n} \alpha_h s^{n-h}$. We also introduce the addition operator + on the space $B$. The unit element of this group $\beta_0 = B(0, 0, \ldots, 0)$ is the mean response of all the treatment combinations. The parameters $\beta_u = B(0, 0, \ldots, \alpha_k, 0, \ldots, 0)$, $k = 1, 2, \ldots, n$, where $\alpha_k \geq 1$ in the $k^{th}$ position corresponds to the $k^{th}$ factor $\alpha_k^{th}$ degree main effect. Interactions correspond to points where coordinates are zero or non-zero with at least two coordinate non-zeros. Later, we also use the following notations: $B_0$ for $B(0, 0, \ldots, 0)$, $B_k^{\alpha_k}$ for $B(0, \ldots, \alpha_k, 0, \ldots, 0)$, and $B_1^{\alpha_1} B_2^{\alpha_2} \ldots B_n^{\alpha_n}$ for $B(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Let $X^{(s)}$ be the matrix of coefficients of orthogonal polynomials of order $s$, where the elements of the first column are all 1 and the inner product of any two different column vectors of $X^{(s)}$ is zero. This matrix $X^{(s)}$ corresponds to a factor level vector $(0, 1, \ldots, s-1)'$. The matrix $X$ can be defined as:

$$X = X^{(s^n)} = X^{(s)} \otimes \ldots \otimes X^{(s)} ,$$

where $\otimes$ denotes the Kronecker product, i.e., if

$$X^{(s)} = \begin{bmatrix} 1 & \xi_{01} & \cdots & \xi_{0,s-1} \\ 1 & \xi_{11} & \cdots & \xi_{1,s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_{p-1,1} & \cdots & \xi_{s-1,s-1} \end{bmatrix},$$

then

$$X^{(s^n)} = \begin{bmatrix} X^{(s^{n-1})} & \xi_{01}X^{(s^{n-1})} & \cdots & \xi_{0,s-1}X^{(s^{n-1})} \\ X^{(s^{n-1})} & \xi_{11}X^{(s^{n-1})} & \cdots & \xi_{1,s-1}X^{(s^{n-1})} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(s^{n-1})} & \xi_{s-1,1}X^{(s^{n-1})} & \cdots & \xi_{s-1,s-1}X^{(s^{n-1})} \end{bmatrix}. \qquad (1.3)$$

Note: Let $\underline{x}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the column vector in $X^{(s^n)}$ corresponding to the parameter $B(\alpha_1, \alpha_2, \cdots, \alpha_n) = B_1^{\alpha_1} B_2^{\alpha_2} \ldots B_n^{\alpha_n}$, and let us also use the notations $\underline{x}(\alpha_h)$, $\underline{x}(\alpha_h, \alpha_k)$, etc. for $B_h^{\alpha_h}$, $B_h^{\alpha_h} B_k^{\alpha_k}$, etc., respectively, and define a specialized product of two matrices $A_{m\times n} = \|a_{ij}\|$ and $B_{m\times n} = \|b_{ij}\|$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, such that

$$A : B = \|c_{ij}\|, \text{ where } c_{ij} = a_{ij}b_{ij}, \quad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n,$$

then, it is easily verified that

$$\underline{x}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \underline{x}(\alpha_1) : \underline{x}(\alpha_2) : \ldots : \underline{x}(\alpha_n) \, . \tag{1.4}$$

Suppose that the vector $y(Z)$ and $\underline{B}$ are rearranged and partitioned as follows: $y(Z^*)' = (y(Z_p)', y(Z_{N-p})')$, $\underline{B}^{*\prime} = (\underline{B}'_p, \underline{B}'_{N-p})$, where $y(Z_p)$ and $\underline{B}_p$ are $p \times 1 = (n(s-1)+1) \times 1$ observation and main effect parameter vectors, respectively, with the mean parameter as the first element of $\underline{B}_p$ and $N = s^n$. We shall write $\underline{y}_p$ and $\underline{y}_{N-p}$ for $y(Z_p)$ and $y(Z_{N-p})$, respectively.

Consider the following expression:

$$E\begin{bmatrix} \underline{y}_p \\ \underline{y}_{N-p} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \cdot \begin{bmatrix} \underline{B}_p \\ \underline{B}_{N-p} \end{bmatrix} \tag{1.5}$$

such that $X_{11}$ is a non-singular $p \times p$ matrix. The existence of $\underline{y}_p$ is easy to verify. From (1.5) we obtain

$$E[\underline{y}_p] = [X_{11}, X_{12}]\,[\underline{B}'_p, \underline{B}'_{N-p}]' , \tag{1.6}$$

and the observations in $\underline{y}_p$ yield a saturated fractional replicate for the parameter vector $\underline{B}_p$.

Using the least squares method (Banerjee and Federer [1964], Zacks [1963]), we obtain the following solution.

$$\hat{\underline{B}}^*_p = \hat{\underline{B}}_p + X_{11}^{-1} X_{12} \hat{\underline{B}}_{N-p} = X_{11}^{-1} \underline{y}_p , \tag{1.7}$$

or

$$\hat{\underline{B}}^*_p = \hat{\underline{B}}_p + (X_1'X_1)^{-1}X_{11}'(I + \lambda\lambda')X_{12}\hat{\underline{B}}_{N-p}$$
$$= (X_1'X_1)^{-1}X_{11}'(I + \lambda\lambda')\underline{Y}_p\,, \tag{1.8}$$

where $X_1 = [X_{11}' X_{21}']'$, and $\lambda = -X_{12}X_{22}^{-1}$. Hence $X_{11}^{-1}\underline{Y}_p$ is the best linear unbiased estimator of $\underline{B}^*_p = \underline{B}_p + X_{11}^{-1}X_{12}\underline{B}_{N-p}$. We note that

$$\text{var}\begin{bmatrix} \hat{\underline{B}}_p \\ \hat{\underline{B}}_{N-p} \end{bmatrix} = \begin{bmatrix} (X_{11}'X_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \sigma^2\,,$$

so that

$$\text{var}(\hat{\underline{B}}_p) = (X_{11}'X_{11})^{-1}\sigma^2\,.$$

## 2. AN INVARIANT PROPERTY OF $|X_{11}'X_{11}|$

First we shall prove the following lemma.

Lemma 2.1. _In an_ $s^n$_-factorial, if_ $X^{(s)}$ _is the corresponding matrix of coefficients of orthogonal polynomials of order_ $s$ _to_ $(0, 1, \ldots, s-1)'$ _and_ $X_1^{(s)}$ _is the matrix corresponding to_ $(1, 2, \ldots, s-1, 0)' = (0, 1, \ldots, s-1)' + (1, 1, \ldots, 1)'$ (mod s), _and_ $X_i^{(s)}$ _is the matrix corresponding to_ $(i, i+1, \ldots, i-1)' = (0, 1, \ldots, s-1)' + (i, i, \ldots, i)'$, (mod s), _then_

(i) _there exist_ $s \times s$ _matrices_ A _and_ B _such that_

$$X_1^{(s)} = AX^{(s)}, \quad X_i^{(s)} = A^i X^{(s)}$$

$$X_1^{(s)} = X^{(s)}B, \quad \underline{\text{and}} \quad X_i^{(s)} = X^{(s)}B^i,$$

(ii) $A^s = I_{s\times s}$ <u>and</u> $B^s = I_{s\times s}$, $|A| = 1$, <u>and</u> $|B| = 1$,

(iii) <u>the matrix</u> B <u>has the form</u>

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdot & \cdots & \cdot \\ \vdots & \vdots & C & \\ 0 & \cdot & & \end{bmatrix}, \tag{2.1}$$

$$C^s = I_{(s-1)\times(s-1)}, \quad \underline{\text{and}} \quad |C| = 1.$$

<u>Proof</u>: Let

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

then, clearly $A^s = I$, $X_1^{(s)} = Ax^{(s)}$, $X_i^{(s)} = A^i X^{(s)}$, $|A^s| = 1$, and $|A| = 1$. Next, letting $X_1^{(s)} = X^{(s)}B$, $B = (X^{(s)})^{-1}X_1^{(s)}$ since $X^{(s)}$ is a non-singular matrix. Then $B = (X^{(s)})^{-1}AX^{(s)} = (X^{(s)})^{-1}X_1^{(s)}$, and $B^i = (X^{(s)})^{-1}A^i X^{(s)} = (X^{(s)})^{-1}X_i^{(s)}$. Hence, $X_i^{(s)} = X^{(s)}B^i$ and $B^s = I_{s\times s}$. We may write B as $(X^{(s)'}X^{(s)})^{-1}X^{(s)'}X_1^{(s)}$ and since $X^{(s)'}X^{(s)}$ is a diagonal matrix and $X^{(s)'}X_1^{(s)}$ has the form

$$\begin{bmatrix} s & 0 & \cdots & 0 \\ 0 & \cdot & \cdots & \cdot \\ \vdots & \cdot & W & \\ 0 & \cdot & & \end{bmatrix},$$

B has the form (2.1), $B^s = I$, $|B| = 1$, $C^s = I_{(s-1)\times(s-1)}$, and $|C| = 1$. This finishes the proof of the lemma.

Let $Z_p(s^n)$ be a saturated main effect plan; write this as $Z_p$, represented by a submatrix of $Z$ such as a $p \times n$ matrix in an $s^n$-factorial and $X_{11}$ by a $p \times p$ coefficient matrix of the main effect parameters corresponding to the plan $Z_p$, and let $J(i_1, i_2, \ldots, i_n)$ be a $p \times n$ matrix such that

$$J(i_1, i_2, \ldots, i_n) = \begin{bmatrix} i_1, i_2, \ldots, i_n \\ \cdots \\ i_1, i_2, \ldots, i_n \end{bmatrix},$$

where $i_h = 0, 1, \ldots, s-1$ for all $h = 1, 2, \ldots, n$, and $X_{11,v}$ be a $p \times p$ coefficient matrix of the main effect parameters corresponding to the plan $Z_{p,v} = Z_p + J(i_1, i_2, \ldots, i_n)$, (mod s), where the order subscript $v = \sum_{h=1}^{n} i_h s^{n-h}$. Then, we obtain the following theorem:

<u>Theorem 1</u>. <u>If</u> $Z_p$ <u>is a saturated main effect plan, then</u> $Z_{p,v}$ <u>also is a saturated main effect plan and</u> $|X'_{11,v}X_{11,v}| = |X'_{11}X_{11}|$.

<u>Proof:</u> From the Lemma 2.1, we obtain

$$X_{11,v} = X_{11} \cdot \mathrm{diag}(1, C^{i_1}, C^{i_2}, \ldots, C^{i_n})$$

where

$$\mathrm{diag}(1, C^{i_1}, \ldots, C^{i_n}) = \begin{bmatrix} 1 & & & & 0 \\ & C^{i_1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & C^{i_n} \end{bmatrix} ,$$

and $C^i$ is the submatrix of $B^i = \left| \begin{array}{c|c} 1 & 0 \\ \hline 0 & C^i \end{array} \right|$ .

Since $|C^{i_n}| = 1$ for $i_h = 0, 1, \ldots, s-1$, $|X_{11,v}| = |X_{11}|$, $|X'_{11,v} X_{11,v}| = |X'_{11} X_{11}|$, and $|X_{11}| \neq 0$, $Z_{p,v}$ also is a saturated main effect plan. This completes the proof of the theorem.

The meaning of this theorem is that if $Z_p$ is not a subgroup (in the algebraic sense) of $Z$ in an $s^n$-factorial, $Z_p + J(i_1, i_2, \ldots, i_n)$, $i_h = 0, 1, \ldots, s-1$ for all $h = 1, 2, \ldots, n$, may produce $s^n$ different main effect plans, but determinants of the information matrices have the same value. (It appears that Webb [1965] is the only one who was aware of the fact that several plans gave the same value for the determinant and that any one was as useful as any other one).

A main effect plan $Z_{p,v}$ of an $s^n$-factorial is said to be <u>independent</u> of a main effect plan $Z_p$ if $Z_{p,v}$ cannot be constructed by the procedure $Z_p + J(i_1, i_2, \ldots, i_n)$, $i_h = 0, 1, \ldots, s-1$ for all $h = 1, 2, \ldots, n$. If

$Z_{p,v}$ and $Z_p$ are not independent then the plan $Z_{p,v}$ is an element of the set, $S(Z_p) = \{Z_p + J(i_1, i_2, \ldots, i_n) : i_h = 0, 1, \ldots, s-1$ for all $h = 1, 2, \ldots, n\}$. The set $S(Z_p)$ is said to be the main effect plan set <u>generated</u> by $Z_p$. Using this criterion, we may list every independent main effect plan from an $s^n$-factorial. We present a complete list of the generators in the cases for $2^2$, $2^3$, and $2^4$-factorials in the Appendix. Since there are $m(s-1)$ main effect parameters in an $s^m$ factorial the total number of main effects plans is $\binom{s^m}{m(s-1)+1}$ and the total number of generators of main effect plan is $\binom{s^m-1}{m(s-1)}/(m(s-1)+1)$ for $m(s-1)+1$ not equal to $s^k$. Thus, for the $2^4$ factorial there are $\binom{15}{4}/5 = 273$ generators as listed in the Appendix.

## 3. REARRANGING THE TREATMENT ORDER AND THE CORRESPONDING DESIGN MATRIX

If we recall the solution (1.7) or (1.8), we note the inverse of $X_{11}$, or of $X_{22}$, is needed to obtain the solution. Also, we see later that if the size of the fraction is less than $s^{n-1}$ in an $s^n$-factorial, then we may use the matrix $X^{(s^{n-1})}$ instead of the $N \times N$ matrix $X^{(s^n)}$ to obtain a solution such as (1.7) or (1.8). Also, we shall see in this case that the method of constructing a saturated fractional replicate resolves itself into the problem of selecting the smallest number of treatments from those corresponding to the orthogonal matrix $X^{(s^{n-k})}$ for some $k \geq 1$. However, in this case, the mean effect will be confounded with the main effect $B_1$. This is the reason for rearranging the treatment order in $Z$ with some higher order defining contrast before constructing a fractional replication, i.e., we shall require the mean effect to be unconfounded with the main effects.

Now consider rearranging the treatment order in vector Z with some defining contrast in an $s^n$-factorial. The $s^n-1$ degrees of freedom among the $s^n$ treatment combinations may be partitioned into $(s^n-1)/(s-1)$ sets of s-1 degrees of freedom. Each set of s-1 degrees of freedom is given by the contrast among s sets of $s^{n-1}$ treatment combinations specified by the following equations:

$$\alpha_1 i_1 + \alpha_2 i_2 + \ldots + \alpha_n i_n = j, \quad j = 0, 1, \ldots, s-1, \bmod s, \tag{3.1}$$

where the right-hand sides of these equations are elements of the Galois Field GF(s). The $\alpha_h$'s are positive integers between 0 and s-1, not all equal to zero, and all additions and multiplications are done within the Galois Field GF(s), then the interaction $B_1^{\alpha_1} B_2^{\alpha_2} \ldots B_n^{\alpha_n}$ corresponds to the equation whose left-hand side is $\alpha_1 i_1 + \alpha_2 i_2 + \ldots + \alpha_n i_n$, i.e., the $(j+1)^{th}$ set of s-1 degrees of freedom given the defining contrast $M \doteq B_1^{\alpha_1} B_2^{\alpha_2} \ldots B_n^{\alpha_n}$, where M denotes the mean parameter, and $\doteq$ means completely confounded with, may be expressed as: $M_j \doteq (B_1^{\alpha_1} B_2^{\alpha_2} \ldots B_n^{\alpha_n})_j$ which satisfies the following condition: $\alpha_1 i_1 + \alpha_2 i_2 + \ldots + \alpha_n i_n = j$, mod s, where $i_h = 0, 1, \ldots, s-1$ for all $h = 1, 2, \ldots, n$. For a defining contrast $M \doteq B_1 B_2^{\alpha_2} \ldots B_n^{\alpha_n}$, the identity relationships are written as:

$$M_j \doteq (B_1 B_2^{\alpha_2} \ldots B_n^{\alpha_n})_j, \quad j = 0, 1, \ldots, s-1. \tag{3.2}$$

Let the set of treatment combinations for fixed $i_1 = \beta$, $\beta = 0, 1, \ldots, s-1$, be $\{\beta, i_2, \ldots, i_n\}$, then the $(k + \beta s^{n-1})^{th}$ treatment corresponds to $M_{j+\beta=t, \bmod s}$, in the set of $\{\beta, i_2, \ldots, i_n\}$.

It is understood that an orthogonal array of strength $d$, of size $N^*$, and with $n$ factors each at $s$ levels, consists of a set of $N^*$ treatment combinations from an $s^n$-factorial arrangement with the property that all $s^d$ treatment combinations corresponding to any d factors chosen from $n$, occur an equal number of times, say $\lambda$ times, in the subset. The orthogonal arrays are denoted by:

$$(N^*, n, s, d, \lambda) .$$

Then it follows that

$$N^* = \lambda s^d .$$

Let $Z(j)$ whose elements are in $Z$, be an $s^{n-1} \times n$ matrix corresponding to $M_j \doteq (B_1 B_2^{\alpha_2} \ldots B_n^{\alpha_n})_j$; then $Z(j)$ is an orthogonal array denoted by

$$(s^{n-1}, n, s, d = \text{at least } 2, \lambda) \tag{3.3}$$

$$\text{for } j = 0, 1, \ldots, s-1 .$$

Theorem 3.1: In an $s^n$-factorial ($s$ is a prime number), if the treatment order in $Z$ is rearranged to correspond to the defining contrasts $M_j \doteq (B_1 B_2^{\alpha_2} \ldots B_n^{\alpha_n})_j$, $j = 0, 1, \ldots, s-1$, then the following form of the corresponding linear orthogonal comparisons matrix $X^*$ can be obtained by rearranging the row vector in $X$, i.e.,

$$X^* = \begin{bmatrix} X^*_{00} & X^*_{01} & \cdots & X^*_{0,s-1} \\ X^*_{00} & X^*_{11} & \cdots & X^*_{1,s-1} \\ \vdots & \vdots & \ddots & \vdots \\ X^*_{00} & X^*_{s-1,1} & \cdots & X^*_{s-1,s-1} \end{bmatrix} \tag{3.4}$$

where $X^*_{00} = X^{(s^{n-1})}$ and $X^*_{ij}$, $j = 0, 1, \ldots, s-1$, are all $s^{n-1} \times s^{n-1}$ matrices.

Proof: Let $X^{(s^{n-1})}_{(\beta)}$ be a matrix corresponding to $\{\beta, i_2, \ldots, i_n\}$ in $X^{(s^n)}$ and let $\{k_t^{(\beta)} = t : t = \sum_{h=2}^{n} i_h s^{n-h-1}\}_j$ be the sequence of the row order numbers in $X^{(s^{n-1})}_{\beta}$ corresponding to $M_j$.

From the fact that $Z(j)$ are orthogonal arrays such as (3.3), each number of elements of the $\{k_t^{(\beta)}\}_j$ is the same for $\beta = 0, 1, \ldots, s-1$. Then the set of sequences

$$\{\{k_t^{(0)}\}_j, \ \{k_t^{(1)}\}_j, \ \ldots, \ \{k_t^{(s-1)}\}_j\} \text{ given } j, \tag{3.5}$$

consists of $s^{n-1}$ non-negative integers less than or equal to $s^{n-1}-1$, and none of the integers is equal to another one. Hence,

$$\{\{k_t^{(0)}\}_j, \{k_t^{(1)}\}_j, \ldots, \{k_t^{(s-1)}\}_j\} = \{\{k_t^{(0)}\}_0, \{k_t^{(0)}\}_1, \ldots, \{k_t^{(0)}\}_{s-1}\}. \quad (3.6)$$

Let $\{\underline{k}^{(\beta)}\}_j$ be the set of the row vectors corresponding to $M_j$ in $X_{(\beta)}^{(s^{n-1})}$, then

$$\begin{bmatrix} \{\underline{k}^{(0)}\}_j \\ \{\underline{k}^{(1)}\}_j \\ \vdots \\ \{\underline{k}^{(s-1)}\}_j \end{bmatrix} \sim \begin{bmatrix} \{\underline{k}^{(0)}\}_0 \\ \{\underline{k}^{(0)}\}_1 \\ \vdots \\ \{\underline{k}^{(0)}\}_{s-1} \end{bmatrix} = X_{(0)}^{(s^{n-1})} \quad ,$$

where the notation $\sim$ means that if we rearrange the row vector order properly in the left-hand side of the matrix of the $\sim$ notation, then this matrix will be the same as $X_{(0)}^{(s^{n-1})}$. This proves the theorem.

<u>Remark:</u> Let

$$\underline{x}_{ij} \; (j, j_2, \ldots, j_n) \quad (3.7)$$

be the column vector in $X_{ij}^*$ corresponding to the parameter $B_1^j B_2^{j_2} \ldots B_n^{j_n}$, where $j_h = 0, 1, \ldots, s-1$ for $h = 2, 3, \ldots, n$. We may obtain the following relations:

$$\underline{x}_{ij}(j, j_2, \ldots, j_n) = \underline{x}_{00}(0, j_2, \ldots, j_n) : \underline{x}_{ij}(j, 0, 0, \ldots, 0)$$

$$= \underline{x}_{00}(0, 0, \ldots, 0) : \underline{x}_{00}(0, j_2, 0, \ldots, 0) : \ldots$$

$$: \underline{x}_{00}(0, 0, \ldots, j_n) : \underline{x}_{ij}(j, 0, \ldots, 0). \qquad (3.8)$$

Theorem 3.2: In an $s^n$-factorial, let $X^*_{0\cdot} = [X^*_{00}, X^*_{01}, \ldots, X^*_{0,s-1}]$ be the $s^{n-1} \times s^n$ matrix corresponding to $Z(0)$ with defining contrast $M_0 \doteq (B_1 B_2^{\alpha_2} \ldots B_n^{\alpha_n})_0$, where at least two of $\alpha_2, \ldots, \alpha_n$ are not zero, then the mean and main effect columns in $X^*_{0\cdot}$ are orthogonal to each other.

Proof: Let $U_{11}$ be a matrix which is constructed using the mean and main effect columns in $X^*_{0\cdot}$ and $\underline{u}_h(j)$ be the column vector corresponding to $B_h^j$ in $U_{11}$, and define $u_0 = \underline{1}$. Since $Z(0)$ is an orthogonal array such as (3.3), (i) in each column of $Z(0)$, each level number occurs an equal number of times, say $\mu$ times, and (ii) all $s^2$ treatment combinations corresponding to any two factors, chosen from $n$ factors, occur an equal number of times, say $\nu$ times, in $Z(0)$.

Then, using a property of $X^{(s)}$, the following relations hold in the matrix $U_{11}$:

$$\underline{u}_0 \cdot \underline{u}_h(j) = \mu \sum_{i=0}^{s-1} \xi_{ij} = 0 \quad \text{for } j = 0, 1, \ldots, s-1;\ h = 1, 2, \ldots, n$$

$$\underline{u}_h(j) \cdot \underline{u}_h(g) = \mu \sum_{i=0}^{s-1} \xi_{ij}\xi_{ig} = 0 \quad \text{for} \quad j \neq g;\ j, g = 0, 1, \ldots, s-1$$

$$\underline{u}_h(j) \cdot u_k(g) = \nu \sum_{i=1}^{s-1} \sum_{m=1}^{s-1} \xi_{ij}\xi_{mg} = 0 \text{ for } h \neq k;\ j, g = 0, 1, \ldots, s-1$$

$$\text{and } h, k = 1, 2, \ldots, n$$

and the theorem is proved.

Theorem 3.3: Let $X^*_{0\cdot} = [X^*_{00}, X^*_{01}]$ be a $2^{n-1} \times 2^n$ matrix corresponding to $Z(0)$ with defining contrast $M_0 \doteq (B_1 B_2^{\alpha_2} \cdots B_n^{\alpha_n})_0$, $\alpha_h = 0$ or 1 for $h = 1, 2, \ldots, n$, in a $2^n$-factorial, then the $X^*_{0\cdot}$ can be rearranged as follows:

$$X_{0\cdot} = [X^*_{00}, \pm X^*_{00}] , \tag{3.9}$$

where the parameter order corresponding to column order in $X_{0\cdot}$ is $M$, $B_n, \ldots, B_2 B_3 \cdots B_n$; $W, B_n W, \ldots, B_2 B_3 \cdots B_n W$, where $W = B_1 B_2^{\alpha_2} \cdots B_n^{\alpha_n}$.

Proof: Using the notation (3.7), the column vectors in $X^*_{0\cdot}$, corresponding to $M, B_n, \ldots, B_2 B_3 \cdots B_n$; $W, B_n W, \ldots, B_2 B_3 \cdots B_n W$ are expressed as $\underline{x}_{00}(0, 0, \ldots, 0)$, $\underline{x}_{00}(0, 0, \ldots, 1), \ldots, \underline{x}_{00}(0, 1, \ldots, 1)$; $\underline{x}_{01}(1, \alpha_2, \ldots, \alpha_n)$, $\underline{x}_{00}(0, 0, \ldots, 1) : \underline{x}_{01}(1, \alpha_2, \ldots, \alpha_n), \ldots, \underline{x}_{00}(0, 1, \ldots, 1) : \underline{x}_{01}(1, \alpha_2, \ldots, a_n)$, respectively.

From the defining contrast $M_0 \doteq (B_1 B_2^{\alpha_2} \cdots B_n^{\alpha_n})_0$,

$$\underline{x}_{01}(1, \alpha_2, \ldots, \alpha_n) = \pm \underline{x}_{00}(0, 0, \ldots, 0), \tag{3.10}$$

where the sign + or − is dependent upon whether $1 + \sum_{h=2}^{n} \alpha_h$ is an even or odd number in this $2^n$-factorial system. Then,

$$\begin{aligned} \underline{x}_{00}(0, 0, \ldots, 1) : \underline{x}_{01}(1, \alpha_2, \ldots, \alpha_n) &= \underline{x}_{00}(0, 0, \ldots, 1) : [\pm \underline{x}_{00}(0, 0, \ldots, 0)] \\ &= \pm \underline{x}_{00}(0, 0, \ldots, 1) \end{aligned}$$

. . .

$$\underline{x}_{00}(0, 1, \ldots, 1) : \underline{x}_{01}(1, \alpha_2, \ldots, \alpha_n) = \pm \underline{x}_{00}(0, 1, \ldots, 1) . \tag{3.11}$$

Using the results in (3.10) and (3.11), we see that this completes the proof of the theorem.

## 4. CONSTRUCTION OF SATURATED FRACTIONAL REPLICATES

We shall consider mostly the method of constructing saturated main effect plans in an $s^n$-factorial. Although we could always construct various saturated non-orthogonal plans for any given parameter set, the general steps of the construction method may not be too instructive. The following steps, however, will be common in constructing any fractional replicate for the specified parameters (also, see Banerjee and Federer [1966] and Paik and Federer [1967] in this connection). Special cases will be illustrated in the following examples.

Step 1. Given the design matrix and parameter and observation vectors, $X\underline{B} = E(\underline{y})$ in any fashion and not necessarily that of the previous section, we now rearrange the parameter matrix such that the $p$ parameters, $p < N$, are arranged to have the $p$ parameters of interest first and $N-p$ parameters not of interest last to obtain $\underline{B}^*$ rearranged $[\underline{B}_p', \underline{B}_{N-p}']'$. This also rearranges the columns of $X$ such that

$$X^*\underline{B}^* = E(\underline{y}) \tag{4.1}$$

or

$$[X_1, X_2] \begin{bmatrix} \underline{B}_p \\ \\ \underline{B}_{N-p} \end{bmatrix} = E(\underline{y}) , \qquad (4.2)$$

where $X^* = [X_1, X_2]$, $X_1$ is an $N \times p$ matrix, and $X_2$ is an $N \times (N-p)$ matrix.

Step 2. Search through rows of $X_1$ until there is an $X_{11}$, $p \times p$, which is non-singular.

Step 3. Corresponding to the rows in $X_{11}$ will be rows in $X_1$ and treatments in Z. Rearrange the treatments in Z into $[Z_p', Z_{N-p}']'$, where $Z_p$ corresponds to the rows in $X_{11}$ from $X_1$. The treatment combinations $Z_p$ yield a saturated design for the parameters in $\underline{B}_p$. This obtained set is one of the possible sets. All possible sets are found by defining all $X_{11}$ which have an inverse.

<u>Example 4.1</u>. Saturated main effect plans in a $2^4$-factorial.

If we consider a $2^4$-factorial design matrix $X^{(2^4)}$ with the defining contrast $M \doteq ABCD$, $(ABCD = B_1B_2B_3B_4$ if we use the notation in section 3), then the alias scheme is as follows:

$$M \doteq ABCD,\ A \doteq BCD,\ B \doteq ACD,\ C \doteq ABD,\ D \doteq ABC,\ AB \doteq CD,\ AC \doteq BD,$$
$$BC \doteq AD .$$

After rearranging the rows and columns taking into consideration the above alias scheme and after using Theorems 3.2 and 3.3, we obtain the following matrix $X^*$:

$$X^{*} = \begin{bmatrix} X^{*}_{00} & X^{*}_{00} \\ X^{*}_{00} & -X^{*}_{00} \end{bmatrix} \tag{4.3}$$

where $X^{*}_{00} = X^{(2^3)}$, and in this case, the treatment order is

$$\begin{array}{l} 0000,\ 0011,\ 0110,\ 0101,\ 1010,\ 1001,\ 1100,\ 1111; \\ 1000,\ 1011,\ 1110,\ 1101,\ 0010,\ 0001,\ 0100,\ \text{and}\ 0111, \end{array} \tag{4.4}$$

and the parameter order is

$$\begin{array}{l} M,\ D,\ C,\ CD,\ B,\ BD,\ BC,\ BCD; \\ ABCD,\ ABC,\ ABD,\ AB,\ ACD,\ AC,\ AD,\ \text{and}\ A. \end{array} \tag{4.5}$$

Now consider the saturated main effect plans in a $2^4$-factorial. Let the treatments be arranged such as (4.4) and using the $7^{th}$, $6^{th}$, and $4^{th}$ column in $X^{*}_{00}$ corresponding to effect BC, BD, and CD, and let $U_{12}$ be an $8 \times 3$ matrix corresponding to parameters BC, BD, CD in $X^{*}_{00}$, then we may easily find three independent rows in the matrix $U_{12}$ and obtain the saturated main effect plans in a $2^4$-factorial.

Let $(n_1, n_2, n_3, n_4, n_5)$, where $n_i$ is a treatment order number in (4.4), be one of the plans constructed by the above procedure, then by recalling theorems (3.1) and (3.3) we know the following treatment combinations are also saturated main effect plans in a $2^4$-factorial, i.e.,

$$(n_1 + 8,\ n_2 + 8,\ n_3 + 8,\ n_4 + 8,\ n_5 + 8). \tag{4.6}$$

Finally, it will be worthwhile to note that all plans (64 plans) in this example belong to the sets generated by the following generators:

$$\begin{array}{cccc} 0000 & 0000 & 0000 & 0000 \\ 0011 & 0011 & 0101 & 0101 \\ 0101 & 1001 & 1001 & 1001 \\ 0110 & 1010 & 1100 & 1111 \\ 1001 & 0101 & 0011 & 0011 \end{array} . \tag{4.7}$$

In these cases, $|X'_{11}X_{11}| = 1024 = 32^2$ .

Example 4.2. Saturated main effect plans in a $3^3$-factorial.

In a $3^3$-factorial, after rearranging the row order for the defining contrast $M \doteq ABC^2$, ($ABC^2 = B_1B_2B_3^2$ if we use the notation in section 3), we obtain the following matrix:

$$X^* = \begin{bmatrix} X^*_{00} & X^*_{01} & X^*_{02} \\ X^*_{00} & X^*_{11} & X^*_{12} \\ X^*_{00} & X^*_{21} & X^*_{22} \end{bmatrix} , \tag{4.8}$$

where each $X^*_{ij}$ is a $9 \times 9$ square matrix, $X^*_{00} = X^{(3^2)}$, and treatment order is

$$\begin{array}{l} 000,\ 011,\ 022,\ 101,\ 112,\ 120,\ 202,\ 210,\ 221; \\ 100,\ 111,\ 122,\ 201,\ 212,\ 220,\ 002,\ 010,\ 021; \\ 200,\ 211,\ 222,\ 001,\ 012,\ 020,\ 102,\ 110,\ \text{and}\ 121, \end{array} \tag{4.9}$$

and the parameter order is

$$M,\ C,\ C^2,\ B,\ BC,\ BC^2,\ B^2,\ B^2C,\ B^2C^2;$$
$$A,\ AC,\ AC^2,\ AB,\ ABC,\ ABC^2,\ AB^2,\ AB^2C,\ AB^2C^2; \qquad (4.10)$$
$$A^2,\ A^2C,\ A^2C^2,\ A^2B,\ A^2BC,\ A^2BC^2,\ A^2B^2,\ A^2B^2C, \text{ and } A^2B^2C^2 .$$

We find that each $X^*_{ij}$ in (4.8) is a non-singular matrix and if we rearrange the column order in $X^*$ to correspond to the following parameter order:

$$M,\ A,\ A^2,\ B,\ B^2,\ C,\ C^2,\ BC,\ BC^2,\ \ldots,$$

and let the first $9 \times 9$ submatrix of the rearranged matrix be $A_{00}$, and if we use the symbols $\underline{M}$, $\underline{A}$, $\underline{A}^2$, $\underline{B}$, $\underline{B}^2$, $\underline{C}$, $\underline{C}^2$, $\underline{BC}$, $\underline{BC}^2$ as the symbol of each corresponding column vector in $A_{00}$, respectively, then, from theorem 3.2, the column vectors $\underline{M}$, $\underline{A}$, $\underline{A}^2$, $\underline{B}$, $\underline{B}^2$, $\underline{C}$, and $\underline{C}^2$ are orthogonal to each other and also $\underline{M}$, $\underline{B}$, $\underline{B}^2$, $\underline{C}$, $\underline{C}^2$, $\underline{BC}$, and $\underline{BC}^2$ are orthogonal to each other. Hence we can say that matrix $\mathbf{A}_{00}$ is non-singular, and then we can make $\underline{BC}$ and $\underline{BC}^2$ orthogonal vectors to the first 7 column vectors. Let such new vectors of $\underline{BC}$ and $\underline{BC}^2$ be $\underline{z}_1$ and $\underline{z}_2$ respectively; then, if we find a non-singular $2 \times 2$ matrix from the $9 \times 2$ matrix, $[\underline{z}_1, \underline{z}_2]$, we can construct a corresponding information matrix $X_{11}$ for saturated main effect plans.

Let $(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$, where $n_i$ is the treatment order number in (4.9), be one of the plans constructed from the above procedure, then the following sets of treatment combinations are also saturated main effect plans in a $3^3$-factorial, i.e.,

$$(n_1 + 9,\ n_2 + 9,\ n_3 + 9,\ n_4 + 9,\ n_5 + 9,\ n_6 + 9,\ n_7 + 9)$$

and

$$(n_1 + 18,\ n_2 + 18,\ n_3 + 18,\ n_4 + 18,\ n_5 + 18,\ n_6 + 18,\ n_7 + 18). \qquad (4.11)$$

In this example, all above plans (81 plans) belong to the sets generated by the following generators:

| | | |
|---|---|---|
| 000 | 000 | 000 |
| 011 | 011 | 011 |
| 022 | 022 | 022 |
| 101 | 101 | 101 |
| 112 | 112 | 120 |
| 120 | 202 | 210 |
| 202 | 210 | 221 |

In these cases, $|X'_{11}X_{11}| = 419904 = 3^2(2^3 \cdot 3^3)^2$ .

Remarks:

(i) In the case of saturated main effect plans in a $2^4$-factorial, every $|X'_{11}X_{11}|$ has one of the four values, i.e., 2304, 1024, 256 or 0. The set generated by a plan (0000, 0111, 1011, 1101, 1110) has the maximum value 2304.

Note that

$$2304 = (3 \cdot 2^4)^2 = 48^2 ,$$

$$1024 = (2 \cdot 2^4)^2 = 32^2 ,$$

$$256 = (1 \cdot 2^4)^2 = 16^2, \text{ and}$$

$$0 = (0 \cdot 2^4)^2 .$$

Also, note that there are 16(1) plans for which $|X_{11}'X_{11}| = 2304$, 16(20) plans for which $|X_{11}'X_{11}| = 1024$, 16(167) plans for which $|X_{11}'X_{11}| = 256$, and 16(85) plans for which $|X_{11}'X_{11}| = 0$.

(ii) In the case of saturated main effect plans in a $3^3$-factorial, every $|X_{11}'X_{11}|$ has one of the five values, i.e., 746496, 419904, 186624, 46656, or 0. The sets generated by the following 9 plans have the maximum value 746496.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 000 | 000 | 000 | 000 | 000 | 000 | 000 | 000 | 000 |
| 021 | 012 | 012 | 011 | 011 | 012 | 011 | 022 | 022 |
| 101 | 102 | 021 | 101 | 102 | 101 | 101 | 202 | 202 |
| 112 | 110 | 102 | 112 | 110 | 110 | 110 | 220 | 220 |
| 120 | 121 | 110 | 120 | 201 | 211 | 122 | 211 | 011 |
| 202 | 201 | 211 | 210 | 121 | 021 | 212 | 121 | 101 |
| 210 | 220 | 220 | 222 | 222 | 222 | 221 | 112 | 110 |

It is of interest to note that for $2^3(3^3) = 216$ that

$$746496 = [4(216)]^2$$

$$419904 = [3(216)]^2$$

$$186624 = [2(216)]^2$$

$$46656 = [1(216)]^2, \text{ and}$$

$$0 = [0(216)]^2 .$$

From the results obtained one is lead to the conjecture that the values of the determinants of $X_{11}$ are $[s(s-1)(s-2)\ldots 1]^m \cdot [m(s-1)-i]$ for $i = s-1, s, s+1, \ldots, m(s-1)$ for saturated main effect plans from an $s^m$-factorial with $m(s-1)+1$ observations, where the number of plans having $|X_{11}|$ equal to a specific value could be zero as in the $2^2$ case.

## 5. ALIAS SCHEMES IN SOME FRACTIONAL REPLICATES

This section is concerned with some alias schemes in some fractional replications. Ehrenfeld and Zacks [1961] and Paik and Federer [1970] presented randomized procedures to obtain an unbiased estimator of $\underline{B}_p$ in place of $\underline{B}_p^* = \underline{B}_p + X_{11}^{-1}X_{12}\underline{B}_{N-p}$ which estimates a sum of parameters. However a randomized procedure may not be always applicable as, for example, in the missing data situation where the data are not missing at random or in situations wherein certain treatments are inadmissible. In such cases, we may want to know the pattern of $X_{11}^{-1}X_{12}$ in irregular fractional replicates as this gives the aliasing scheme.

5.1. Alias schemes in saturated fractional replicates for the $2^4$-factorial.

In example 4.1 (saturated main effect plans in a $2^4$-factorial), suppose that the following partitioned matrix of $X$ is obtained after rearranging the columns in $X^*$ (the row order was arranged subject to $M \doteq ABCD$) in (4.3).

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \left[\begin{array}{c|cc} X_{11} & X_{1211} & X_{1212} \\ \hline X_{2111} & X_{2211} & X_{2212} \\ X_{2121} & X_{2221} & X_{2222} \end{array}\right], \tag{5.1}$$

where the parameter order corresponding in $X$ is as follows: M, A, B, C, D, CD, BD, BC; ABCD, BCD, ACD, ABD, ABC, AB, AC, AD, and $X_{11}$ is a $p \times p$ $(p < 8)$ non-singular matrix, $X_{2111}$ and $X'_{1211}$ are each $p \times (8-p)$ matrices, $X_{2211}$ is an $(8-p) \times (8-p)$ matrix, $X_{2121}$ and $X'_{1212}$ are $8 \times p$ matrices, and $X_{2221}$ and $X'_{2212}$ are $8 \times (8-p)$ matrices.

We know from (4.3) that

$$\begin{bmatrix} X_{11} & X_{1211} \\ X_{2111} & X_{2211} \end{bmatrix} = \begin{bmatrix} X_{1212} \\ X_{2212} \end{bmatrix} \sim X^*_{00} ,$$

so that

$$X_{1212} = [X_{11}, X_{1211}] .$$

Then, $X_{12}$ can be partitioned as follows:

$$X_{12} = [X_{1211}, X_{11}, X_{1211}] . \tag{5.2}$$

Hence,

$$X_{11}^{-1}X_{12} = [X_{11}^{-1}X_{1211}, \ I, \ X_{11}^{-1}X_{1211}] \ . \tag{5.3}$$

It may be easily verified that, in all plans in the example 4.1 (there are 64 plans), $X_{11}^{-1}X_{12}$ has the following form:

$$[\pm \hat{X}_{11}^{-1}X_{1211} \ , \ \pm I, \ \pm X_{11}^{-1}X_{1211}] . \tag{5.4}$$

Example 5.1.

1000
1011
1101
1110
0001

This plan may be obtained by the following procedure:

$$\begin{bmatrix} 0000 \\ 0011 \\ 0101 \\ 0110 \\ 1001 \end{bmatrix} + \begin{bmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1011 \\ 1101 \\ 1110 \\ 0001 \end{bmatrix}, \ \text{mod } 2 \ .$$

In this case, we obtain the following solution:

$$\hat{\underline{B}}_p + \left[\begin{array}{ccccc|ccc|ccc} -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \hat{\underline{B}}_{N-p} = X_{11}^{-1}\underline{y}_p \ ,$$

where $\underline{B}'_p = (M, A, B, C, D)$ and $\hat{\underline{B}}'_{N-p} = $ (ABCD, BCD, ACD, ABD, ABC; AB, AC, BC; CD, BD, AD).

Similar results may be obtained from the plans which are constructed by the method of example 4.1 with defining contrasts $M \doteq ABC$, $M \doteq ABD$, $M \doteq ACD$, $M \doteq BCD$. (In all these cases, $|X'_{11}X_{11}| = 1024$). None of the saturated main effect plans except the above plans with $|X'_{11}X_{11}| = 1024$ in the $2^4$-factorial has the form in (5.4).

For the $2^n$-factorial system, saturated fractions with $|X'_{11}X_{11}|$ a maximum do not always have the best aliasing structure for $X_{11}^{-1}X_{12}$ given that complete confounding of effects has better properties than having an effect in $\underline{B}_p$ partially confounded with all the parameters in $\underline{B}_{N-p}$.

<u>The case of</u> $p > 8$ <u>for a</u> $2^4$<u>-factorial</u>.

In a $2^4$-factorial, suppose that $M \doteq ABCD$, $\underline{B}'_p = (M, A, B, C, D, AB, AC, AD, BC)$, and the matrix $X$ in (5.1) is partitioned as follows:

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} X_{00} & X_{0012} & X_{121} \\ X_{0021} & X_{0022} & X_{122} \\ \hline X_{221} & X_{222} & X_{22} \end{array}\right] \qquad (5.5)$$

where the parameter order corresponding to the columns in $X$ is M, A, B, C, D, AB, AC, AD, BC; ABCD, BCD, ACD, ABD, ABC, CD, BD, and $X_{11}$ is a $p \times p$ non-singular matrix, $X_{22}$ is an $(N-p) \times (N-p)$ non-singular matrix, and $X_{00} \sim X^*_{00}$ in (4.3). Then, the treatment designation of observations in the order corresponding to the matrix $X$ is 0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111, and 1101, and the remaining 7 treatments in some order.

Since

$$X^{-1} = X' = \begin{bmatrix} X'_{11} & X'_{21} \\ X'_{12} & X'_{22} \end{bmatrix},$$

$$X'_{11}X_{12} = -X'_{21}X_{22}$$

$$X'_{12}X_{12} = I_{(N-p)\times(N-p)} - X'_{22}X_{22} .$$

Since

$$X_{11}^{-1} = X'_{11} - X'_{21}X'^{-1}_{22}X'_{12} ,$$

then

$$\begin{aligned} X_{11}^{-1}X_{12} &= X'_{11}X_{12} - X'_{21}X'^{-1}_{22}X'_{12}X_{12} \\ &= -X'_{21}X_{22} - X'_{21}X'^{-1}_{22}(I - X'_{22}X_{22}) \\ &= -X'_{21}X'^{-1}_{22} . \end{aligned}$$

Also, since the matrix $X_{21}$ may be partitioned as

$$X_{21} = [-X_{22} \;\; -X_{222} \;\; X_{222}] \, ,$$

$$X_{11}^{-1}X_{12} = -\begin{bmatrix} -X'_{22} \\ \\ -X'_{222} \\ \\ X'_{222} \end{bmatrix} X'^{-1}_{22}$$

$$= \begin{bmatrix} I_{(N-p)\times(N-p)} \\ \\ X'_{222}X'^{-1}_{22} \\ \\ -X'_{222}X'^{-1}_{22} \end{bmatrix} \, , \qquad (5.6)$$

Example 5.2. (p = 9)

Suppose that the treatment combinations corresponding to $X_{11}$ in (5.5) are 0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111, and 1101. Then

$$X_{11}^{-1}X_{12} = \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 \end{array}\right] \, ,$$

where $\underline{B}'_p$ = (M, A, B, C, D, AB, AC, AD, BC) and $\underline{B}'_{N-p}$ = (ABCD, BCD, ACD, ABD, ABC, CD, BD).

If $p = 12$, we may find a fractional plan in a $2^4$-factorial such that

$$X'_{222} X'^{-1}_{22} = -I_{4\times 4} ,$$

for example, in the case of (0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1100, 1101, 1110, 1111),

$$X_{11}^{-1} X_{12} = \begin{bmatrix} I_{4\times 4} \\ -I_{4\times 4} \\ I_{4\times 4} \end{bmatrix}$$

where $\underline{B}'_p$ = (M, D, C, CD, B, BD, BC, BCD, A, AD, AC, ACD) and $\underline{B}'_{N-p}$ = (AB, ABD, ABC, ABCD).

5.2. Some unsaturated main effect fractional replicates.

In the saturated main effect plans in an $s^n$-factorial, $s > 2$, we are unable to find the pattern of $X_{11}^{-1} X_{12}$ similar to the cases in a $2^n$-factorial, because, in general, $X^*_{ij} \neq cX^*_{i'j'}$, where $c$ is some constant, for $i \neq i'$ and $j \neq j'$ in an $s^n$-factorial $(s > 2)$. However, we shall present an application of a similar method to the case in example 5.1 in some unsaturated main effect fractional replicates in an $s^n$-factorial $(s > 2)$.

Consider the following split-plot type design in a $3^3$-factorial:

| | | | |
|---|---|---|---|
| 000 | 100 | 200 | |
| 001 | 101 | 201 | |
| 010 | 110 | 210 | ; |
| 012 | 112 | 212 | |
| 022 | 122 | 222 | |

then, from (1.3), we may obtain the following three equations:

$$\hat{\underline{B}}_5 + [X_{11}^{-1}X_{1211},\ \ -I,\ \ -X_{11}^{-1}X_{1211},\ \ I,\ \ X_{11}^{-1}X_{1211}]\hat{\underline{B}}_{22} = X_{11}^{-1}\underline{y}_{5,1},$$

$$\hat{\underline{B}}_5 + [X_{11}^{-1}X_{1211},\ \ (0)I,\ \ (0)X_{11}^{-1}X_{1211},\ \ -2I,\ \ -2X_{11}^{-1}X_{1211}]\hat{\underline{B}}_{22} = X_{11}^{-1}\underline{y}_{5,2},$$

$$\hat{\underline{B}}_5 + [X_{11}^{-1}X_{1211},\ \ I,\ \ X_{11}^{-1}X_{1211},\ \ I,\ \ X_{11}^{-1}X_{1211}]\hat{\underline{B}}_{21} = X_{11}^{-1}\underline{y}_{5,3}, \qquad (5.7)$$

where $\underline{B}_5 = (M, C, C^2, B, B^2)'$, $B_{22} = (\underline{B}_4', \underline{B}_{5,a}', \underline{B}_{4,a}', \underline{B}_{5,a^2}', \underline{B}_{4,a^2}')'$, where $\underline{B}_4 = (BC, BC^2, B^2C, B^2C^2)'$, $\underline{B}_{5,a} = (A, AC, AC^2, AB, AB^2)'$, $\underline{B}_{4,a} = (ABC, ABC^2, AB^2C, AB^2C^2)'$, $\underline{B}_{5,a^2} = (A^2, A^2C, A^2C^2, A^2B, A^2B^2)'$, and $\underline{B}_{4,a^2} = (A^2BC, A^2BC^2, A^2B^2C, A^2B^2C^2)'$, and $X_{11}$ is a $5 \times 5$ matrix, $X_{1211}$ is a $5 \times 4$ matrix, $I$ is a $5 \times 5$ identity matrix, and $\underline{y}_{5,1}$, $\underline{y}_{5,2}$, and $\underline{y}_{5,3}$ are observation vectors.

From (5.7), we obtain:

$$\hat{\underline{B}}_5 + X_{11}^{-1}X_{1211}\hat{\underline{B}}_4 = \frac{1}{3}X_{11}^{-1}(\underline{y}_{5,1} + \underline{y}_{5,2} + \underline{y}_{5,3})$$

$$\hat{\underline{B}}_{5,a} + X_{11}^{-1}X_{1211}\hat{\underline{B}}_{4,a} = \frac{1}{2}X_{11}^{-1}(\underline{y}_{5,3} - \underline{y}_{5,1})$$

$$\hat{\underline{B}}_{5,a^2} + X_{11}^{-1}X_{1211}\hat{\underline{B}}_{4,a^2} = \frac{1}{6}X_{11}^{-1}(\underline{y}_{5,1} - 2\underline{y}_{5,2} + \underline{y}_{5,3}).$$

In the above, an aliasing structure property was mentioned in connection with the examples. The goodness of the aliasing structure property is determined by the number of effects that are partially or completely confounded with each other. The fewer the number of effects confounded with each other the more desirable is the aliasing structure property, that is, the more nearly the aliasing structure is to complete confounding of effects the more desirable it is. Likewise, the greater the number of effects partially confounded with each other the more undesirable is the plan. The fewer the number of effects that are confounded with any specified effect, the larger will be the number of effects that are orthogonal to the effect. Now, in order to completely describe the aliasing structure property, it is necessary to have an ordering of patterned matrices from a diagonal matrix to nonzero submatrices on the diagonal with zeros elsewhere, to submatrices which form diagonal matrices and on down to a matrix with no zero elements. Perhaps some classification of the aliasing matrix $X_{11}^{-1}X_{12}$ could be made on the number or proportion of zero elements in the matrix. When this problem is resolved, the aliasing property structure with its criterion for goodness will be completely described.

It should be pointed out that the aliasing structure property described above may be more appropriate in many experimental situations than is the minimum variance (maximum value of the determinant of $X_{11}'X_{11}$) property. Hence a fractional replicate may result in a maximum value of $|X_{11}'X_{11}|$ but may have an undesirable aliasing structure. In this case, a plan for which $|X_{11}'X_{11}|$ is not maximum would be selected in preference to one for which $|X_{11}'X_{11}|$ was maximum.

# APPENDIX

## Classifications of the Saturated Main Effect Plans in $2^2$, $2^3$, and $2^4$-factorials

1. Generator of the saturated main effect plans in a $2^2$-factorial. There is only one generator and $|X'_{11}X_{11}| = 16 = [2^2]^2$

$$\begin{array}{c} 00 \\ 01 \\ 10 \end{array} \left\{ \begin{bmatrix} 00 \\ 01 \\ 10 \end{bmatrix} + \begin{bmatrix} 00 \\ 00 \\ 00 \end{bmatrix} = \begin{bmatrix} 00 \\ 01 \\ 10 \end{bmatrix}, \begin{bmatrix} 00 \\ 01 \\ 10 \end{bmatrix} + \begin{bmatrix} 01 \\ 01 \\ 01 \end{bmatrix} = \begin{bmatrix} 01 \\ 00 \\ 11 \end{bmatrix}, \begin{bmatrix} 00 \\ 01 \\ 10 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 00 \end{bmatrix}, \begin{bmatrix} 00 \\ 01 \\ 10 \end{bmatrix} + \begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \\ 01 \end{bmatrix} \right\}$$

2. Generators of the saturated main effect plans in a $2^3$-factorial.

(i) $|X'_{11}X_{11}| = 256 = [2 \cdot 2^3]^2$

000*
011
101
110

(ii) $|X'_{11}X_{11}| = 64 = [1 \cdot 2^3]^2$

| | | | | | | |
|---|---|---|---|---|---|---|
| 000 | 000 | 000 | 000 | 000 | 000 | 000 |
| 100 | 100 | 100 | 001 | 101 | 011 | 011 |
| 101 | 101 | 110 | 101 | 110 | 101 | 110 |
| 110 | 111 | 111 | 111 | 111 | 111 | 111 |

(iii) $|X'_{11}X_{11}| = 0 = [0 \cdot 2^3]^2$

| | | | | | |
|---|---|---|---|---|---|
| 000* | 000* | 000* | 000* | 000* | 000* |
| 001 | 001 | 010 | 001 | 010 | 011 |
| 010 | 100 | 100 | 110 | 101 | 100 |
| 011 | 101 | 110 | 111 | 111 | 111 |

*A regular fraction of a $2^3$-factorial. (This yields only two plans instead of $2^3 = 8$ plans.)

3. Generators of the saturated main effect plans in a $2^4$-factorial.

(i) $|X'_{11}X_{11}| = 2304$

0000
0111
1011
1101
1110

(ii) $|X'_{11}X_{11}| = 1024$

| M $\doteq$ ABCD | | | | M $\doteq$ ABC | | | |
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| 0000 | 0000 | 0000 | 0000 | 0000 | 0000 | 0000 | 0000 |
| 0011 | 0011 | 0011 | 0011 | 0001 | 0001 | 0001 | 0001 |
| 0101 | 0101 | 0101 | 0101 | 0110 | 0110 | 0110 | 0110 |
| 0110 | 1001 | 1001 | 1001 | 1010 | 1010 | 1011 | 1011 |
| 1001 | 1010 | 1100 | 1111 | 1100 | 1101 | 1100 | 1101 |

$M \doteq ABD$

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| 0101 | 0101 | 0101 | 0101 |
| 1001 | 1001 | 1011 | 1011 |
| 1100 | 1110 | 1100 | 1110 |

$M \doteq ACD$

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| 0100 | 0100 | 0100 | 0100 |
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| 1010 | 1110 | 1101 | 1110 |

$M \doteq BCD$

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| 0011 | 0011 | 0011 | 0011 |
| 0101 | 0101 | 0101 | 0101 |
| 0110 | 1000 | 1011 | 1101 |
| 1000 | 1110 | 1110 | 1110 |

(iii) $|X'_{11}X_{11}| = 256$

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| 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 |
| 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

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| 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 |
| 0101 | 0101 | 0101 | 0101 | 0101 | 0101 | 0101 | 0101 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

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| 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 |
| 0110 | 0110 | 0110 | 0110 | 0110 | 0110 | 0110 | 0110 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

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| 0111 | 0111 | 0111 | 0111 | 0111 | 0111 | 0111 | 0111 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

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| 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 |
| 1000 | 1000 | 1000 | 1000 | 1001 | 1001 | 1001 | 1001 |
| 1100 | 1101 | 1110 | 1111 | 1100 | 1101 | 1110 | 1111 |

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| 1010 | 1010 | 1010 | 1010 | 1011 | 1011 | 1011 | 1011 |
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| 0110 | 0110 | 0110 | 0110 | 0110 | 0110 | 0110 | 0110 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

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| 0111 | 0111 | 0111 | 0111 | 0111 | 0111 | 0111 | 0111 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

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| 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 |
| 1000 | 1000 | 1000 | 1000 | 1001 | 1001 | 1001 | 1001 |
| 1010 | 1011 | 1110 | 1111 | 1010 | 1011 | 1110 | 1111 |

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| 1010 | 1010 | 1011 | 1011 | 1100 | 1100 | 1101 | 1101 |
| 1100 | 1101 | 1100 | 1101 | 1110 | 1111 | 1110 | 1111 |

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| 0110 | 0110 | 0110 | 0110 | 0110 | 0110 | 0110 | 0110 |
| 1000 | 1000 | 1000 | 1000 | 1001 | 1001 | 1001 | 1001 |
| 1010 | 1011 | 1100 | 1101 | 1010 | 1011 | 1100 | 1101 |

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| 1010 | 1010 | 1011 | 1011 | 1100 | 1100 | 1101 | 1101 |
| 1110 | 1111 | 1110 | 1111 | 1110 | 1111 | 1110 | 1111 |

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| 1010 | 1010 | 1010 | 1010 | 1011 | 1011 | 1011 | 1011 |
| 1100 | 1101 | 1110 | 1111 | 1100 | 1101 | 1110 | 1111 |

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| 1000 | 1000 | 1000 | 1000 | 1010 | 1010 | 1010 | 1010 |
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| 1110 | 1111 | 1110 | 1111 | 1110 | 1111 | 1110 | 1111 |

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| 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 |
| 0111 | 0111 | 0111 | 0111 | 0111 | 0111 | 0111 | 0111 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

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| 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 |
| 1000 | 1000 | 1000 | 1001 | 1001 | 1001 | 1010 | 1010 |
| 1011 | 1101 | 1111 | 1010 | 1100 | 1110 | 1101 | 1111 |

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| 0100 | 0100 | 0100 | 0100 | 0101 | 0101 | 0101 | 0101 |
| 1011 | 1011 | 1100 | 1101 | 1000 | 1000 | 1000 | 1001 |
| 1100 | 1110 | 1111 | 1110 | 1011 | 1100 | 1110 | 1010 |

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| 0101 | 0101 | 0101 | 0101 | 0101 | 0101 | 0101 | 0101 |
| 1001 | 1001 | 1010 | 1010 | 1011 | 1011 | 1100 | 1101 |
| 1101 | 1111 | 1100 | 1110 | 1101 | 1111 | 1111 | 1110 |

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| 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1001 | 1001 |
| 1011 | 1011 | 1011 | 1011 | 1100 | 1101 | 1100 | 1101 |
| 1100 | 1101 | 1110 | 1111 | 1111 | 1110 | 1111 | 1110 |

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| 0011 | 0011 | 0011 | 0011 | 0011 | 0011 | 0011 | 0011 |
| 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 |
| 1000 | 1000 | 1001 | 1001 | 1010 | 1010 | 1011 | 1011 |
| 1101 | 1110 | 1100 | 1111 | 1100 | 1111 | 1101 | 1110 |

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| 0101 | 0101 | 0101 | 0101 | 0101 | 0101 | 0101 |
| 1000 | 1000 | 1001 | 1010 | 1010 | 1011 | 1011 |
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(iv) $|X'_{11}X_{11}| = 0$

$M \doteq A$

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$M \doteq B$

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| 0011 | 1000 | 1000 | 1000 | 1001 | 1001 | 1010 |
| 1000 | 1001 | 1010 | 1011 | 1010 | 1011 | 1011 |

$M \doteq C$

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| 0101 | 1000 | 1000 | 1000 | 1001 | 1001 | 1100 |
| 1000 | 1001 | 1100 | 1101 | 1100 | 1101 | 1101 |

$M \doteq D$

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| 1000 | 1010 | 1100 | 1110 | 1100 | 1110 | 1110 |

$M \doteq AB$

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| 1100 | 1101 | 1110 | 1111 | 1110 | 1111 | 1111 |

$M \doteq AC$

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| 0101 | 1010 | 1010 | 1010 | 1011 | 1011 | 1110 |
| 1010 | 1011 | 1110 | 1111 | 1110 | 1111 | 1111 |

M $\doteq$ AD

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| 0010 | 0010 | 0010 | 0010 | 0010 | 0010 | 0010 |
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| 0110 | 1001 | 1001 | 1001 | 1011 | 1011 | 1101 |
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M $\doteq$ BC

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| 1000 | 1001 | 1110 | 1111 | 1110 | 1111 | 1111 |

M $\doteq$ BD

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| 0101 | 0101 | 0101 | 0101 | 0101 | 0101 | 0101 |
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| 1000 | 1010 | 1101 | 1111 | 1101 | 1111 | 1111 |

M $\doteq$ CD

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| 0100 | 0100 | 0100 | 0100 | 0100 | 0100 | 0100 |
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| 1000 | 1011 | 1100 | 1111 | 1100 | 1111 | 1111 |

$M \doteq ABC$

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$M \doteq ABD$

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$M \doteq ABCD$

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| 0101 | 0101 | 0101 |
| 1010 | 1010 | 1100 |
| 1100 | 1111 | 1111 |

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13. ABSTRACT

A generalized method of constructing irregular fractional replicates from a complete factorial is developed. An invariance property and an aliasing structure property of fractional replicate plans are presented.

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